

Special 2-cocycles and 3–3 Pachner move relations in Grassmann algebra

Igor G. Korepanov

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Abstract

Grassmann-algebraic relations, corresponding naturally to Pachner move 3–3 in four-dimensional topology, are presented. They involve 2-cocycles of two specific forms, and some more homological objects.

1 Introduction

This short note presents a construction of a 3–3 Pachner move relation in Grassmann algebra — the four-dimensional analogue of pentagon relation known in three-dimensional algebraic topology. Below:

- in Section 2, the Grassmann–Berezin calculus of anticommuting variables is briefly recalled,
- in Section 3, Pachner move 3–3 is explained, together with a possible form of algebraic relation corresponding to it,
- in Section 4, actual four-simplex Grassmann weights are presented satisfying this relation, and a cocyclic property of their coefficients is stated,
- in Section 5, a generalization of our Grassmann weights is proposed involving new coefficients, also, apparently, of homological nature,
- and in Appendix on page 6, different Grassmann weights are presented, apparently — according to numerical evidence — satisfying the same 3–3 relation.

2 Grassmann–Berezin calculus

A *Grassmann algebra* over a field \mathbb{F} of characteristic > 2 is an associative \mathbb{F} -algebra with unity, generators x_i and relations

$$x_i x_j = -x_j x_i.$$

In particular, $x_i^2 = 0$, so an element of a Grassmann algebra is a polynomial of degree ≤ 1 in each x_i . If it consists only of monomials of even (odd) total degree, it is called an even (odd) element.

The *exponent* is defined by its Taylor series. For instance,

$$\exp(x_1x_2 + x_3x_4) = 1 + x_1x_2 + x_3x_4 + x_1x_2x_3x_4.$$

The *Berezin integral* [1] in a variable (= generator) x_i is, by definition, the \mathbb{F} -linear operator

$$f \mapsto \int f \, dx_i$$

in Grassmann algebra satisfying

$$\int dx_i = 0, \quad \int x_i dx_i = 1, \quad \int gh \, dx_i = g \int h \, dx_i,$$

if g does not contain x_i ; multiple integral is understood as iterated one, according to the following model:

$$\iint xy \, dy \, dx = \int x \left(\int y \, dy \right) dx = 1.$$

3 Pachner move 3–3 and a proposed form of algebraic relation

Pachner moves [8] are elementary local rebuildings of a manifold triangulation. A triangulation of a piecewise-linear manifold can be transformed into another triangulation using a finite sequence of Pachner moves, see [6] for a pedagogical introduction.

There are five (types of) Pachner moves in four dimensions, of which move 3–3 is, in some informal sense, central. It transforms a cluster of three four-simplices situated around a two-face into a cluster of three other four-simplices situated around another two-face, and occupying the same place in the manifold. We say that these clusters form the left- and right-hand sides of Pachner move, respectively. There are six vertices in each cluster, we denote them $1 \dots 6$, and the four-simplices will be 12345, 12346 and 12356 in the l.h.s., and 12456, 13456 and 23456 in the r.h.s. Thus, the common inner two-face is 123 in the l.h.s., and 456 in the r.h.s.

An algebraic relation whose l.h.s. and r.h.s. can be said to correspond naturally to the l.h.s. and r.h.s. of a Pachner move gives hope of constructing an invariant of piecewise-linear manifolds.

Remark. Four other Pachner moves in four-dimensional topology are $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$. Experience shows that if an interesting formula related to move 3–3 has been discovered, then there are also formulas corresponding to other moves.

The Grassmann-algebraic Pachner move relations proposed in this paper have the following form:

$$\begin{aligned} f_{123} \int \mathcal{W}_{12345} \mathcal{W}_{12346} \mathcal{W}_{12356} dx_{1234} dx_{1235} dx_{1236} \\ = \pm f_{456} \iiint \mathcal{W}_{12456} \mathcal{W}_{13456} \mathcal{W}_{23456} dx_{1456} dx_{2456} dx_{3456}. \end{aligned} \quad (1)$$

Here Grassmann variables x_{ijkl} are attached to all three-faces $ijkl$; the *Grassmann weight* \mathcal{W}_{ijklm} of a four-simplex $ijklm$ depends on (i.e., contains) the variables on its three-faces, e.g., \mathcal{W}_{12345} depends on x_{1234} , x_{1235} , x_{1245} , x_{1345} and x_{2345} . The integration goes in variables on *inner* three-faces in the corresponding side of Pachner move, while the result depends on the variables on boundary faces. Also, there are numeric factors f_{ijk} before the integrals, thought of as attached to the respective inner two-faces $ijk = 123$ or 456 .

4 Grassmann weights satisfying the 3–3 relation, and a cocyclic property of coefficients

We now present Grassmann four-simplex weights \mathcal{W}_{ijklm} and factors f_{ijk} , satisfying the 3–3 algebraic relation (1). These will depend on the *coordinates* of vertices: we attach to each vertex i two numbers $\xi_i, \eta_i \in \mathbb{F}$ that must be generic enough so that the expressions (4) below never vanish.

Remark. Or we can take *indeterminates* over \mathbb{F} — algebraically independent elements — for ξ_i and η_i .

We define \mathcal{W}_{ijklm} as the following Grassmann–Gaussian exponent:

$$\mathcal{W}_{ijklm} = \exp \Phi_{ijklm}, \quad (2)$$

where

$$\Phi_{ijklm} = p_{ijklm} \sum_{\substack{\text{over 2-faces } abc \\ \text{of } ijklm}} \epsilon_{d_1abcd_2}^{ijklm} \varphi_{abc} x_{\{abcd_1\}} x_{\{abcd_2\}}, \quad (3)$$

and below we explain the notations in (3).

First, both p_{ijklm} and $\epsilon_{d_1abcd_2}^{ijklm}$ are signs. The first of them reflects the consistent orientation of four-simplices, namely, for the left-hand side

$$p_{12345} = 1, \quad p_{12346} = -1, \quad p_{23456} = 1,$$

and for the right-hand side

$$p_{12456} = 1, \quad p_{13456} = -1, \quad p_{23456} = 1.$$

As for the epsilon, it is the sign of permutation between the sequences of its subscripts and superscripts.

Second, the value φ_{abc} is defined as follows:

$$\varphi_{abc} = \begin{vmatrix} 1 & 1 & 1 \\ \xi_a & \xi_b & \xi_c \\ \eta_a & \eta_b & \eta_c \end{vmatrix}. \quad (4)$$

Thus, φ_{abc} belongs to an *oriented* two-face: for instance, $\varphi_{abc} = -\varphi_{bac}$.

And third, the curly brackets in (3) serve to emphasize that the Grassmann variable $x_{\{abcd\}}$ does *not* depend on the order of indices a, b, c, d .

Theorem 1. *The weights \mathcal{W}_{ijklm} defined in this Section satisfy the relation (1), with*

$$f_{ijk} = \frac{1}{\varphi_{ijk}},$$

and the sign before the right-hand side is minus.

Proof. Direct calculation. I used our package PL [3] for manipulations in Grassmann algebra. \square

Statement. *Values φ_{abc} form a 2-cocycle: for a tetrahedron $abcd$,*

$$\varphi_{bcd} - \varphi_{acd} + \varphi_{abd} - \varphi_{abc} = 0.$$

Proof. Simple calculation using (4). \square

5 A generalization involving still more homological objects

We are now going to generalize our Grassmann four-simplex weights using still more objects of, apparently, homological nature.

Calculations show that the exponent (2) has actually no terms of degree > 2 , that is,

$$\exp \Phi_{ijklm} = 1 + \Phi_{ijklm}. \quad (5)$$

We now change the definition (2) to the following:

$$\mathcal{W}_{ijklm} = h_{ijklm} + \Phi_{ijklm}, \quad (6)$$

where h_{ijklm} is some numeric coefficient (or, more generally, an even element of Grassmann algebra).

Theorem 2. *The relation (1) holds also for weights defined according to (6), provided the coefficients h_{ijklm} in its r.h.s. are expressed through those in its l.h.s. as follows:*

$$\begin{aligned}\varphi_{456} h_{12456} &= \varphi_{345} h_{12345} - \varphi_{346} h_{12346} + \varphi_{356} h_{12356} , \\ \varphi_{456} h_{13456} &= \varphi_{245} h_{12345} - \varphi_{246} h_{12346} + \varphi_{256} h_{12356} , \\ \varphi_{456} h_{23456} &= \varphi_{145} h_{12345} - \varphi_{146} h_{12346} + \varphi_{156} h_{12356} .\end{aligned}$$

Proof. Direct calculation. □

There appears to be analogy between the constructions in this paper and those in [4]. Recall that second homologies, in their exotic form, do enter in Grassmanian 3–3 relations in the mentioned paper. Our present relations look substantially simpler, which may be important for constructing a TQFT. Their homological nature is still to be clarified.

The 3–3 relations proposed here have been found while trying to generalize the pentagon relation in [5] to four-dimensional case.

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Appendix: A relation with a quadratic form of rank 4

The explanation of relation (5) for the Grassmanian quadratic form Φ_{ijklm} introduced in Section 4 lies in the fact that Φ_{ijklm} has *rank 2*. A generic Grassmanian quadratic form of five variables has, however, rank 4 — the maximal rank of an antisymmetric 5×5 matrix. It is natural to expect (for instance, from a comparison with the pentagon relation in [5]) that relations with quadratic forms of rank 4 will expose richer mathematical structure than those of rank 2.

In this Appendix, I report new Grassmann weights \mathcal{W}_{ijklm} , with Φ_{ijklm} of rank 4. They are constructed almost exactly as in Section 4, except that φ_{abc} is defined not by (4), but as follows:

$$\varphi_{abc} = \frac{\xi_a - \xi_b}{1 + \xi_a \xi_b} \cdot \frac{\xi_b - \xi_c}{1 + \xi_b \xi_c} \cdot \frac{\xi_c - \xi_a}{1 + \xi_c \xi_a}. \quad (7)$$

Here, the ξ_i are again vertex coordinates — indeterminates over \mathbb{F} , or such values that $\varphi_{abc} \neq 0, \infty$. No more coordinates, like η_i in Section 4, are used here.

Remark. Formula (7) can of course be written more elegantly in terms of tangent function over such fields \mathbb{F} where tangent is well defined.

Experimental result. *The weights \mathcal{W}_{ijklm} with φ_{abc} defined according to (7) satisfy the same relation (1), refined according to the same Theorem 1.*

Also, the values (7) have the same cocyclic property as described in the Statement after Theorem 1.

These new Grassmann weights were found by guess-and-try method combined with some theoretical ideas that are to be disclosed later. By now, I was only able to check numerically the validity of relation (1) for these weights.